

# A NON-SELF-ADJOINT LEBESGUE DECOMPOSITION

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**ABSTRACT.** We study the structure of bounded linear functionals on a class of non-self-adjoint operator algebras that includes the multiplier algebra of every complete Nevanlinna-Pick space, and in particular the multiplier algebra of the Drury-Arveson space. Our main result is a Lebesgue decomposition expressing every linear functional as the sum of an absolutely continuous (i.e. weak-\* continuous) linear functional, and a singular linear functional that is far from being absolutely continuous. This is a non-self-adjoint analogue of Takesaki's decomposition theorem for linear functionals on von Neumann algebras. We apply our decomposition theorem to prove that the predual of every algebra in this class is (strongly) unique.

## 1. INTRODUCTION

The main result in this paper is a decomposition theorem for bounded linear functionals on a class of operator algebras that includes the multiplier algebra of every complete Nevanlinna-Pick space. Results of this kind can be seen as a noncommutative generalization of the Yosida-Hewitt decomposition of a measure into completely additive and purely finitely additive parts, or more classically, the Lebesgue decomposition of a measure into absolutely continuous and singular parts.

Takesaki proved in [Tak58] that a bounded linear functional on a von Neumann algebra can be decomposed uniquely into the sum of a normal (i.e. weak-\* continuous) linear functional, and a singular linear functional that is far from being normal. In [And78], Ando proved a direct analogue of Takesaki's decomposition theorem for linear functionals on the algebra  $H^\infty$ , of bounded analytic functions on the complex unit disk  $\mathbb{D}$ . More recently, in [Ued09], Ueda proved a generalization of Ando's result for finite maximal subdiagonal algebras, which are “analytic” subalgebras of finite von

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Neumann algebras introduced by Arveson in [Arv67] as a noncommutative generalization of the algebra  $H^\infty$ .

A compelling case can be made that the natural function-theoretic generalization of  $H^\infty$  is the algebra  $H_d^\infty$  of multipliers on the Drury-Arveson space  $H_d^2$ . The algebra  $H_d^\infty$  is contained in the algebra  $H^\infty(\mathbb{B}_d)$  of bounded analytic functions on the complex unit ball  $\mathbb{B}_d$  of  $\mathbb{C}^d$ , but for  $d \geq 2$  this inclusion is proper, and  $H_d^\infty$  is seemingly much more tractable than  $H^\infty(\mathbb{B}_d)$  (see for example [Arv98]). The Drury-Arveson space  $H_d^2$  and the multiplier algebra  $H_d^\infty$  are universal in the following sense: Every irreducible complete Nevanlinna-Pick space embeds into  $H_d^2$ , and the corresponding multiplier algebra arises as the compression of  $H_d^\infty$  onto this embedding (see [AM00] for details). Examples of complete Nevanlinna-Pick spaces include the Hardy space and the Dirichlet space on the disk, the Drury-Arveson space itself, and more generally the class of Besov-Sobolev spaces on  $\mathbb{B}_d$ .

One explanation for the tractability of  $H_d^\infty$  is the fact that  $H_d^\infty$  arises as a quotient of the noncommutative analytic Toeplitz algebra  $F_d^\infty$  (see for example [DP98b] and [AP00]). This algebra, introduced by Popescu in [Pop89], can be viewed as an algebra of noncommutative analytic functions acting by left multiplication on a Hardy space  $F_d^2$  of noncommutative analytic functions. The operator-algebraic structure of  $F_d^\infty$ , which is now well understood, turns out to be strikingly similar to that of  $H^\infty$  (see for example [Pop89, Pop95, AP00] and [DP98a, DP98b, DP99]).

For a weak-\* closed two-sided ideal  $\mathcal{I}$  of  $F_d^\infty$ , we let  $\mathcal{A}_\mathcal{I}$  denote the algebra  $\mathcal{A}_\mathcal{I} = F_d^\infty / \mathcal{I}$ . These algebras are the main objects of interest in this paper, for the following reason: The multiplier algebra of every irreducible complete Nevanlinna-Pick space arises as the compression of  $F_d^\infty$  to a coinvariant subspace, and this compression is completely isometrically isomorphic and weak-\* to weak-\* homeomorphic to a quotient of  $F_d^\infty$  by a two-sided ideal (see [DP98b, AP00] for details).

Our main result is the following decomposition theorem for linear functionals on quotients of  $F_d^\infty$ . A functional is said to be absolutely continuous if it is weak-\* continuous, and singular if it is, roughly speaking, far from being weak-\* continuous (we give a precise definition below).

**Theorem 1.1** (Lebesgue decomposition for quotients of  $F_d^\infty$ ). *Let  $\mathcal{I}$  be a weak-\* closed two-sided ideal of  $F_d^\infty$ , and let  $\phi$  be a bounded linear functional on  $\mathcal{A}_\mathcal{I}$ . Then there are unique linear functionals  $\phi_a$  and  $\phi_s$  on  $\mathcal{A}_\mathcal{I}$  such that  $\phi = \phi_a + \phi_s$ , where  $\phi_a$  is absolutely continuous and  $\phi_s$  is singular, and such*

that

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

If  $d = 1$ , then the constant  $\sqrt{2}$  can be replaced with the constant 1. Moreover, these constants are optimal.

The following result for multiplier algebras of complete Nevanlinna-Pick spaces is an immediate consequence of Theorem 1.1.

**Corollary 1.2** (Lebesgue decomposition for multiplier algebras). *Let  $\mathcal{A}$  be the multiplier algebra of a complete Nevanlinna-Pick space, and let  $\phi$  be a bounded linear functional on  $\mathcal{A}$ . Then there are unique linear functionals  $\phi_a$  and  $\phi_s$  on  $\mathcal{A}$  such that  $\phi = \phi_a + \phi_s$ , where  $\phi_a$  is absolutely continuous and  $\phi_s$  is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

We first prove that Theorem 1.1 holds for  $F_d^\infty$ . The proof for quotients of  $F_d^\infty$  requires the following generalization of the classical F. & M. Riesz theorem, which is similar in spirit to the noncommutative F. & M. Riesz-type theorems proved by Exel in [Exe90] for operator algebras with the Dirichlet property, and by Blecher and Labuschagne in [BL07] for maximal subdiagonal algebras.

**Theorem 1.3** (Extended F. & M. Riesz Theorem). *Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ , and let  $\phi = \phi_a + \phi_s$  be the Lebesgue decomposition of  $\phi$  into absolutely continuous and singular parts as in Theorem 1.1. Let  $\mathcal{I}$  be a weak-\* closed two-sided ideal of  $F_d^\infty$ . If  $\phi$  is zero on  $\mathcal{I}$ , then  $\phi_a$  and  $\phi_s$  are both zero on  $\mathcal{I}$ .*

Grothendieck proved in [Gro55] that  $L^1$  is the unique predual of  $L^\infty$  (up to isometric isomorphism). Soon after, in [Sak56], Sakai generalized Grothendieck's result by proving that the predual of every von Neumann algebra is unique. In fact, this latter result follows from the proof of Sakai's characterization of von Neumann algebras as C\*-algebras which are dual spaces.

The uniqueness of the predual of a von Neumann algebra can also be proved using Takesaki's decomposition theorem from [Tak58] (see for example the proof of Corollary 3.9 of [Tak02]). A similar idea was used by Ando in [And78], to prove the uniqueness of the predual of  $H^\infty$ , and more recently, by Ueda in [Ued09], to prove that the predual of every maximal subdiagonal algebra is unique.

Inspired by these results, we apply Theorem 1.3 to prove that the predual of every quotient  $\mathcal{A}_{\mathcal{I}}$  is (strongly) unique.

**Theorem 1.4.** *Let  $\mathcal{I}$  be a weak- $*$  closed two-sided ideal of  $F_d^\infty$ . Then the algebra  $\mathcal{A}_{\mathcal{I}}$  has a strongly unique predual.*

It follows immediately from Theorem 1.4 that the multiplier algebra of every complete Nevanlinna-Pick space has a unique predual.

**Corollary 1.5.** *The multiplier algebra of every complete Nevanlinna-Pick space has a strongly unique predual.*

In particular, Corollary 1.5 implies that the multiplier algebra  $H_d^\infty$  on the Drury-Arveson space has a unique predual. We believe this result is especially interesting in light of the fact that, for  $d \geq 2$ , the uniqueness of the predual of  $H^\infty(\mathbb{B}_d)$  is an open problem.

In addition to this introduction, this paper has five other sections. In Section 2, we provide a brief review of the requisite background material. In Section 3, we prove the Lebesgue decomposition for  $F_d^\infty$ , and give an example showing that the constant in the statement of the theorem is optimal. In Section 4, we prove the extended F. & M. Riesz Theorem. In Section 5, we prove the Lebesgue decomposition theorem for quotients of  $F_d^\infty$ , and hence for multiplier algebras of complete Nevanlinna-Pick spaces. In Section 6, we use the Lebesgue decomposition theorem to prove that the predual of every quotient of  $F_d^\infty$  is unique, and hence that the predual of the multiplier algebra of every complete Nevanlinna-Pick space is unique.

## 2. PRELIMINARIES

**2.1. The noncommutative analytic Toeplitz algebra.** For fixed  $1 \leq d \leq \infty$ , let  $\mathbb{C}\langle Z \rangle = \mathbb{C}\langle Z_1, \dots, Z_d \rangle$  denote the algebra of noncommutative polynomials in the variables  $Z_1, \dots, Z_d$ . As a vector space,  $\mathbb{C}\langle Z \rangle$  is spanned by the set of monomials

$$\{Z_w = Z_{w_1} \cdots Z_{w_n} \mid w = w_1 \cdots w_n \in \mathbb{F}_d^*, n \geq 0\},$$

where  $\mathbb{F}_d^*$  denotes the free semigroup generated by  $\{1, \dots, d\}$ . The noncommutative Hardy space  $F_d^2$  is the Hilbert space obtained by completing  $\mathbb{C}\langle Z \rangle$  in the natural inner product

$$\langle Z_w, Z_{w'} \rangle = \delta_{w,w'}, \quad w, w' \in \mathbb{F}_d^*.$$

Equivalently,  $F_d^2$  is the Hilbert space consisting of noncommutative power series with square summable coefficients,

$$F_d^2 = \left\{ \sum_{w \in \mathbb{F}_d^*} a_w Z_w \mid \sum_{w \in \mathbb{F}_d^*} |a_w|^2 < \infty \right\}.$$

We think of the elements of  $F_d^2$  as noncommutative analytic functions.

Every element in  $F_d^2$  gives rise to a multiplication operator on  $F_d^2$  in the following way (note that in this noncommutative setting, it is necessary to specify whether multiplication occurs on the left or the right). For  $F$  in  $F_d^2$ , the left multiplication operator  $L_F$  is defined by

$$L_F G = FG, \quad G \in H_d^2.$$

The operator  $L_F$  is not necessarily bounded in general, simply because the product of two elements in  $F_d^2$  is not necessarily contained in  $F_d^2$ . However, it is always densely defined on  $\mathbb{C}\langle Z \rangle$ .

The noncommutative analytic Toeplitz algebra  $F_d^\infty$  is the noncommutative multiplier algebra of  $F_d^2$ . It consists precisely of the functions  $F$  in  $H_d^2$  such that the corresponding left multiplication operator is bounded,

$$F_d^\infty = \{F \in H_d^2 \mid FG \in H_d^2, \forall G \in H_d^2\}.$$

Equivalently, if we identify  $F$  in  $F_d^\infty$  with the left multiplication operator  $L_F$  on the Hilbert space  $F_d^2$ , then  $F_d^\infty$  is obtained as the closure of  $\mathbb{C}\langle Z \rangle$  in the weak-\* topology on  $\mathcal{B}(F_d^2)$ . The noncommutative disk algebra  $A_d$  is the closure of  $\mathbb{C}\langle Z \rangle$  in the norm topology. Note that it is properly contained in  $F_d^\infty$ .

The algebras  $A_d$  and  $F_d^\infty$  were introduced by Popescu in [Pop96] and [Pop95] respectively. For  $d = 1$ ,  $F_d^2$  can be identified with the classical Hardy space  $H^2$ ,  $F_d^\infty$  can be identified with the classical algebra of bounded analytic functions, and  $A_d$  can be identified with the classical disk algebra of functions that are analytic on  $\mathbb{D}$  with continuous extensions to the boundary.

## 2.2. The structure of an isometric tuple.

**Definition 2.1.** Let  $V = (V_1, \dots, V_d)$  be an isometric tuple. Then

- (1)  $V$  is a *unilateral shift* if it is unitarily equivalent to a multiple of  $L_Z = (L_{Z_1}, \dots, L_{Z_d})$ ,

- (2)  $V$  is *absolutely continuous* if the unital weak operator closed algebra  $W(V_1, \dots, V_d)$  generated by  $V_1, \dots, V_d$  is algebraically isomorphic to the noncommutative analytic Toeplitz algebra  $F_d^\infty$ ,
- (3)  $V$  is *singular* if the weakly closed algebra  $W(V_1, \dots, V_d)$  is a von Neumann algebra, and
- (4)  $V$  is of *dilation type* if it has no summand that is absolutely continuous or singular.

The next result is from [Ken12].

**Theorem 2.2** (Lebesgue-von Neumann-Wold Decomposition). *Let  $V = (V_1, \dots, V_d)$  be an isometric  $d$ -tuple. Then  $V$  can be decomposed as*

$$V = V_u \oplus V_a \oplus V_s \oplus V_d,$$

where  $V_u$  is a unilateral  $d$ -shift,  $V_a$  is an absolutely continuous unitary  $d$ -tuple,  $V_s$  is a singular unitary  $d$ -tuple and  $V_d$  is a unitary  $d$ -tuple of dilation type.

The next result is from [DKP01].

**Theorem 2.3** (Structure Theorem for Free Semigroup Algebras). *Let  $V = (V_1, \dots, V_d)$  be an isometric  $d$ -tuple, and let  $\mathcal{V} = W(V_1, \dots, V_d)$  denote the unital weak operator closed algebra generated by  $V_1, \dots, V_d$ . Then there is a maximal projection  $P$  in  $\mathcal{V}$  with the range of  $P$  coinvariant for  $\mathcal{V}$  such that*

- (1)  $\mathcal{V}P = \cap_{k \geq 1} (\mathcal{V}_0)^k$ , where  $(\mathcal{V}_0)^k$  denotes the ideal  $(\mathcal{V}_0)^k = \sum_{|w|=k} V_w \mathcal{V}$ ,
- (2) if  $P^\perp \neq 0$ , then the restriction of  $\mathcal{V}$  to the range of  $P^\perp$  is an analytic free semigroup algebra,
- (3) the compression of  $\mathcal{V}$  to the range of  $P$  is a von Neumann algebra, and
- (4)  $\mathcal{V} = P^\perp \mathcal{V} P^\perp + W^*(V)P$ .

Let  $V = V_u \oplus V_a \oplus V_s \oplus V_d$  be the Lebesgue-von Neumann-Wold decomposition of an isometric tuple  $V$ , as in Theorem 2.2, where  $V_u$  is a unilateral  $n$ -shift,  $V_a$  is an absolutely continuous unitary  $n$ -tuple,  $V_s$  is a singular unitary  $n$ -tuple and  $V_d$  is a unitary  $n$ -tuple of dilation type. Suppose that  $V$  is defined on a Hilbert space  $H$ , and let  $H = H_u \oplus H_a \oplus H_s \oplus H_d$  denote the corresponding decomposition of  $H$ . By Corollary 2.7 of [DKP01], there is a maximal invariant subspace  $K$  for  $V_d$  such that the restriction of  $V_d$  to  $K$  is analytic. The projection  $P$  in Theorem 2.3 is determined by  $P^\perp = P_{H_u} \oplus P_{H_a} \oplus P_K$ .

*Remark 2.4.* For  $d = 1$ , an isometry is the direct sum of a unilateral shift, an absolutely continuous unitary and a singular unitary. Theorem 2.3 implies that, in this case, the structure projection  $P$  is the projection onto the singular unitary part. In particular, this implies that  $P$  is reducing. For  $d \geq 2$ , the proof of Theorem 3.3 shows that  $P$  is reducing if and only if there is no summand of dilation type.

**2.3. The universal representation.** We require the universal representation  $\pi_u : F_d^\infty \rightarrow \mathcal{B}(H_u)$  of  $F_d^\infty$ . This can be constructed as in 2.4.4 of [BL05], as the restriction of the universal representation of  $C_{\max}^*(F_d^\infty)$ . By 3.2.12 of [BL05], we can identify the double dual  $(F_d^\infty)^{**}$  of  $F_d^\infty$  with the algebra obtained as the weak-\* closure of  $\pi_u(F_d^\infty)$ . We will require the operator algebra structure on  $(F_d^\infty)^{**}$  provided by this identification. By replacing  $\pi_u$  by  $\pi_u^{(\infty)}$  if necessary, we can suppose that  $\pi_u$  has infinite multiplicity, and hence that the weak operator topology coincides with the weak-\* topology on  $(F_d^\infty)^{**}$ .

Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . By the Hahn-Banach Theorem, we can extend  $\phi$  to a functional on  $C_{\max}^*(F_d^\infty)$  with the same norm. Hence by the construction of the universal representation of  $C_{\max}^*(F_d^\infty)$ , there are vectors  $x$  and  $y$  in  $H_u$  with  $\|x\|\|y\| = \|\phi\|$  such that

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

If we identify  $F_d^\infty$  with its image  $\pi_u(F_d^\infty)$  in  $(F_d^\infty)^{**}$ , then the functional  $\phi$  has a unique weak-\* continuous extension to a functional on  $(F_d^\infty)^{**}$  with the same norm. We will use this fact repeatedly.

Since  $\pi_u$  is the restriction of a \*-homomorphism of  $C_{\max}^*(F_d^\infty)$ , and since the  $d$ -tuple  $(L_{Z_1}, \dots, L_{Z_d})$  is isometric, it follows that the  $d$ -tuple  $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$  is also isometric. Since  $(F_d^\infty)^{**}$  contains  $\pi_u(A_d)$ , it necessarily contains the weak operator closed algebra generated by  $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$ . Let  $P_u$  denote the projection in  $(F_d^\infty)^{**}$  guaranteed by Theorem 2.3. We will refer to  $P_u$  as the universal structure projection in  $(F_d^\infty)^{**}$ .

*Remark 2.5.* Let  $\mathcal{S}$  denote the unital weak operator closed algebra generated by  $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$ . From above we have  $\mathcal{S} \subseteq (F_d^\infty)^{**}$ , and one might guess that  $\mathcal{S} = (F_d^\infty)^{**}$ . However, this is not the case. Indeed, let  $\phi$  be a bounded nonzero functional on  $F_d^\infty$  that is zero on the noncommutative disk algebra  $A_d$ . Then as above, there are vectors  $x$  and  $y$  in  $H_u$  such that

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

Let  $\psi$  denote the weak operator continuous functional on  $\mathcal{S}$  defined by

$$\psi(S) = \langle Sx, y \rangle, \quad \forall S \in \mathcal{S}.$$

Since  $\phi$  is zero on  $A_d$ ,  $\psi$  must be zero on  $\pi_u(A_d)$ . Then, since  $\pi_u(A_d)$  is weak operator dense in  $\mathcal{S}$ , it follows that  $\psi(S) = \langle Sx, y \rangle = 0$  for all  $S$  in  $\mathcal{S}$ . But, by assumption, there is  $A$  in  $F_d^\infty$  such that  $\phi(A) = \langle \pi_u(A)x, y \rangle \neq 0$ . So we see that  $\pi_u(A) \notin \mathcal{S}$ , and hence that the inclusion  $\mathcal{S} \subseteq (F_d^\infty)^{**}$  is proper.

### 3. THE LEBESGUE DECOMPOSITION

In this section, we introduce the definitions of absolutely continuous and singular linear functionals on the noncommutative analytic Toeplitz algebra  $F_d^\infty$ , and establish the first version of the Lebesgue decomposition. In [DLP05], Davidson, Li and Pitts proved a Lebesgue-type decomposition for functionals on the noncommutative disk algebra  $A_d$ . Although the algebra  $F_d^\infty$  is bigger than  $A_d$ , the next definitions is closely related to (and directly inspired by) the corresponding definition for  $A_d$ .

**Definition 3.1.** Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . Then

- (1)  $\phi$  is *absolutely continuous* if it is weak-\* continuous, and
- (2)  $\phi$  is *singular* if  $\|\phi\| = \|\phi^k\|$  for every  $k \geq 1$ , where  $\phi^k$  denotes the restriction of  $\phi$  to the ideal of  $F_d^\infty$  generated by  $\{L_{Z_w} \mid |w| = k\}$ .

Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . Then as in Section 2.3, there are vectors  $x$  and  $y$  in  $H_u$  with  $\|x\|\|y\| = \|\phi\|$  such that

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

We will write  $\phi P_u$  and  $\phi P_u^\perp$  for the linear functionals defined on  $F_d^\infty$  by

$$\begin{aligned} (\phi P_u)(A) &= \langle \pi_u(A)P_u x, y \rangle, \quad \forall A \in F_d^\infty, \\ (\phi P_u^\perp)(A) &= \langle \pi_u(A)P_u^\perp x, y \rangle, \quad \forall A \in F_d^\infty, \end{aligned}$$

where  $P_u$  denotes the universal structure projection from Section 2.3. The purpose of the next result is to verify that  $\phi P_u$  and  $\phi P_u^\perp$  are well defined.

**Lemma 3.2.** *Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . Then the functionals  $\phi P_u$  and  $\phi P_u^\perp$ , as defined above, do not depend on the choice of vectors  $x$  and  $y$ .*

*Proof.* Let  $x_1, y_1$  and  $x_2, y_2$  be pairs of vectors in  $H_u$  such that

$$\langle \pi_u(A)x_1, y_1 \rangle = \langle \pi_u(A)x_2, y_2 \rangle, \quad \forall A \in F_d^\infty.$$



Since  $\pi_u(F_d^\infty)$  is weak-\* dense in the algebra  $(F_d^\infty)^{**}$ , which contains  $P_u$ , it follows immediately that

$$\langle \pi_u(A)P_u x_1, y_1 \rangle = \langle \pi_u(A)P_u x_2, y_2 \rangle, \quad \forall A \in F_d^\infty,$$

and similarly that

$$\langle \pi_u(A)P_u^\perp x_1, y_1 \rangle = \langle \pi_u(A)P_u^\perp x_2, y_2 \rangle, \quad \forall A \in F_d^\infty.$$

□

**Proposition 3.3.** *A bounded functional  $\phi$  on  $F_d^\infty$  is singular if and only if  $\phi = \phi P_u$ .*

*Proof.* Let  $\phi$  be a singular functional on  $F_d^\infty$ . We can assume that  $\|\phi\| = 1$ . As in Section 2.3, there are vectors  $x$  and  $y$  in  $H_u$  such that  $\|x\|\|y\| = 1$  and

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

By the singularity of  $\phi$ , we can find a sequence  $(A_k)$  of elements in  $F_d^\infty$  such that  $\lim \phi(A_k) \rightarrow 1$ , and such that each  $A_k$  belongs to the unit ball of  $(F_{d,0}^\infty)^k = \sum_{|w|=k} F_d^\infty L_{Z_w}$ . Let  $T$  be an accumulation point of the sequence  $(\pi_u(A_k))$  in  $(F_d^\infty)^{**}$ , and let  $\mathcal{S}$  denote the unital weak operator closed algebra generated by  $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$ . It is clear that the weak-\* closure of the image  $\pi_u((F_{d,0}^\infty)^k)$  of the ideal  $(F_{d,0}^\infty)^k$  can be written as  $(F_d^\infty)^{**} \mathcal{S}_0^k$ , where  $\mathcal{S}_0$  denotes the ideal in  $\mathcal{S}$  generated by  $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$ . Thus  $\pi_u(A_k)$  belongs to  $(F_d^\infty)^{**} \mathcal{S}_0^k$ . By Theorem 2.3,  $\mathcal{S}P_u = \bigcap_{k \geq 1} \mathcal{S}_0^k$ . Hence  $T$  belongs to the unit ball of

$$\bigcap_{k \geq 1} (F_d^\infty)^{**} \mathcal{S}_0^k = (F_d^\infty)^{**} \bigcap_{k \geq 1} \mathcal{S}_0^k = (F_d^\infty)^{**} P_u.$$

In particular, this means that  $T = TP_u$ . Since  $\phi(T) = 1$ , this gives

$$\|x\|\|y\| = 1 = \langle Tx, y \rangle = \langle TP_u x, y \rangle \leq \|P_u x\|\|y\| \leq \|x\|\|y\|.$$

Hence  $P_u x = x$ , and it follows that  $\phi = \phi P_u$ .

Conversely, let  $\phi$  be a functional on  $F_d^\infty$  such that  $\phi = \phi P_u$ . As before, we can assume that  $\|\phi\| = 1$ , and there are vectors  $x$  and  $y$  in  $H_u$  such that  $\|x\|\|y\| = 1$  and

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

The fact that  $\phi P_u = \phi$  implies that we can choose  $x$  satisfying  $x = P_u x$ , and hence that

$$\phi(A) = \langle \pi_u(A)P_u x, y \rangle, \quad \forall A \in F_d^\infty.$$

Let  $\psi$  denote the functional on  $(F_d^\infty)^{**}$  defined by

$$\psi(T) = \langle TP_u x, y \rangle, \quad \forall T \in (F_d^\infty)^{**},$$

and for  $k \geq 1$ , let  $\psi^k$  denote the restriction of  $\psi$  to  $(F_d^\infty)^{**}\mathcal{S}_0^k$ . Then as above,

$$(F_d^\infty)^{**}P_u = \bigcap_{k \geq 1} (F_d^\infty)^{**}\mathcal{S}_0^k.$$

Hence  $\|\psi\| = \|\psi^k\|$  for every  $k \geq 1$ . It follows that  $\|\phi\| = \|\phi^k\|$ , where  $\phi^k$  is defined as in Definition 3.1, and hence that  $\phi$  is singular.  $\square$

**Lemma 3.4.** *The range of the projection  $P_u^\perp$  is invariant for  $(F_d^\infty)^{**}$ .*

*Proof.* It suffices to show that whenever  $x$  and  $y$  are vectors in  $F_d^2$  such that  $x = P_u^\perp x$  and  $y = P_u y$ , and the functional  $\phi$  on  $F_d^\infty$  is defined by

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty,$$

then  $\phi = 0$ . By Theorem 2.3, the range of  $P_u^\perp$  is invariant for  $\pi_u(A_d)$ . Hence  $\phi$  is zero on  $A_d$ . Let  $A$  be an element of  $F_d^\infty$ . By Corollary 2.6 of [DP98a], for  $k \geq 1$ , we can write  $A$  uniquely as

$$A = \sum_{|w| < k} a_w L_{Z_w} + A',$$

where  $A'$  belongs to  $(F_{d,0}^\infty)^k$ . The fact that  $\phi$  is zero on  $A_d$  implies that  $\phi(A) = \phi(A')$ . It follows from Definition 3.1 that  $\phi$  is singular. Hence by Proposition 3.3,  $\phi = \phi P_u$ , i.e.

$$\phi(A) = \langle \pi_u(A)P_u x, y \rangle, \quad \forall A \in F_d^\infty.$$

Since  $x = P_u^\perp x$ , it follows that  $\phi = 0$ , as required.  $\square$

**Proposition 3.5.** *Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . Then  $\phi$  is absolutely continuous if and only if  $\phi = \phi P_u^\perp$ .*

*Proof.* Suppose first that  $\phi$  is absolutely continuous. Then it is weak-\* continuous, so there are sequences of vectors  $(x_k)$  and  $(y_k)$  in  $F_d^2$  such that

$$\phi(A) = \sum \langle Ax_k, y_k \rangle, \quad \forall A \in F_d^\infty.$$

Since the  $d$ -tuple  $(L_{Z_1}, \dots, L_{Z_d})$  is equivalent to a restriction of the unilateral shift part of the  $d$ -tuple  $(\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d}))$ ,  $F_d^2$  can be identified with a subspace of  $H_u$ , and it follows that  $\phi = \phi P_u^\perp$ .

Conversely, suppose that  $\phi = \phi P_u^\perp$ . As in Section 2.3, there are vectors  $x$  and  $y$  in  $H_u$  with  $\|x\|\|y\| = \|\phi\|$  such that

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

The fact that  $\phi = \phi P_u^\perp$  implies that we can choose  $x$  satisfying  $P_u^\perp x = x$ . Since, by Lemma 3.4, the range of  $P_u^\perp$  is invariant for  $\pi_u(F_d^\infty)$ , it follows that for every  $A$  in  $F_d^\infty$ , we have

$$\phi(A) = \langle \pi_u(A)x, y \rangle = \langle P_u^\perp \pi_u(A) P_u^\perp x, y \rangle = \langle \pi_u(A) P_u^\perp x, P_u^\perp y \rangle.$$

Hence we can also choose  $y$  satisfying  $P_u^\perp y = y$ .

By the construction of  $P_u$ , the restriction of the operators  $\pi_u(L_{Z_1}), \dots, \pi_u(L_{Z_d})$  to the cyclic subspace generated by  $x$  and  $y$  is analytic. Thus, by the main result of [Ken12], the weak-\* closed algebra generated by this restriction is completely isometrically isomorphic and weak-\* to weak-\* homeomorphic to  $F_d^\infty$ . It follows that  $\phi$  is weak-\* continuous on  $F_d^\infty$ .  $\square$

**Theorem 3.6** (Lebesgue Decomposition for  $F_d^\infty$ ). *Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . Then there are unique linear functionals  $\phi_a$  and  $\phi_s$  on  $F_d^\infty$  such that  $\phi = \phi_a + \phi_s$ , where  $\phi_a$  is absolutely continuous and  $\phi_s$  is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

*If  $d = 1$ , then the constant  $\sqrt{2}$  can be replaced with the constant 1.*

*Proof.* As in Section 2.3, there are vectors  $x$  and  $y$  in  $H_u$  such that  $\|x\|\|y\| = \|\phi\|$  and

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

Define  $\phi_a$  and  $\phi_s$  by  $\phi_a = \phi P_u^\perp$  and  $\phi_s = \phi P_u$  respectively. Then  $\phi_a$  is absolutely continuous by Proposition 3.5, and  $\phi_s$  is singular by Proposition 3.3. We clearly have  $\phi = \phi_a + \phi_s$ . To see that  $\phi_a$  and  $\phi_s$  are unique, suppose that

$$\phi_a + \phi_s = \psi_a + \psi_s,$$

where  $\psi_a$  is absolutely continuous and  $\psi_s$  is absolutely continuous. Then

$$\phi_a - \psi_a = \psi_s - \phi_s.$$

It is clear that the functional  $\phi_a - \psi_a$  is absolutely continuous, and Proposition 3.3 implies that the functional  $\psi_s - \phi_s$  is singular. Applying Proposition

3.5 and Proposition 3.3 again, we can therefore write

$$\phi_a - \psi_a = (\phi_a - \psi_a)P_u^\perp = (\psi_s - \phi_s)P_u^\perp = (\psi_s - \phi_s)P_u P_u^\perp = 0.$$

Hence  $\phi_a = \psi_a$ , and it follows similarly that  $\phi_s = \psi_s$ . Finally, we compute

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \|Px\|\|y\| + \|P^\perp x\|\|y\| \leq \sqrt{2}\|x\|\|y\| = \sqrt{2}\|\phi\|.$$

If  $d = 1$ , then Remark 2.4 implies that  $(F_d^\infty)^{**}$  is the direct sum of two algebras reduced by  $P_u$ . If we identify  $F_d^\infty$  with its image in  $(F_d^\infty)^{**}$ , then the functionals  $\phi$ ,  $\phi_a$  and  $\phi_s$  extend uniquely to weak-\* continuous functionals on  $(F_d^\infty)^{**}$  with the same norm. Since  $\phi_a = \phi_a P_u^\perp$  and  $\phi_s = \phi_s P_u$ , it follows that in this case,  $\|\phi\| = \|\phi_a\| + \|\phi_s\|$ .  $\square$

The next example is based on Example 5.10 from [DLP05]. It establishes that for  $d \geq 2$ , the constant  $\sqrt{2}$  in the statement of Theorem 3.6 is the best possible.

**Example 3.7.** Define  $\phi$  on  $\mathbb{C}\langle Z \rangle$  by setting

$$\phi(L_{Z_w}) = \begin{cases} 1/\sqrt{2} & \text{if } w = \emptyset \text{ or } w = 21^n \text{ for } n \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and extending by linearity. We will first show that  $\phi$  extends to a bounded linear functional on the noncommutative disk algebra  $A_2$ . Let  $\mathcal{H}_\phi$  denote the Hilbert space  $\mathbb{C}e \oplus F_2^2$ , and define a 2-tuple  $S = (S_1, S_2)$  on  $\mathcal{H}_\phi$  by setting

$$S_1 = \begin{pmatrix} I & 0 \\ 0 & L_1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 \\ \xi_\emptyset e^* & L_2 \end{pmatrix}.$$

It is easy to check that  $S$  is isometric. By the universal property of the noncommutative disk algebra, we obtain a completely isometric representation  $\pi_\phi$  of  $A_2$  satisfying

$$\pi_\phi(L_{Z_w}) = S_{w_1} \cdots S_{w_n}, \quad w = w_1 \cdots w_n \in \mathbb{F}_d^*,$$

and we can extend  $\phi$  to  $A_2$  by

$$\phi(A) = \langle \pi_\phi(A)(e + \xi_\emptyset)/\sqrt{2}, \xi_\emptyset \rangle, \quad A \in A_2.$$

From this, it is easy to check that  $\|\phi\| \leq 1$ .

Let  $\mathcal{S}$  denote the unital weakly closed algebra generated by  $S_1$  and  $S_2$ . The structure projection from Theorem 2.3 is the projection  $P$  onto  $\mathbb{C}e$ , which is contained in  $\mathcal{S}$ . Hence  $\mathcal{S}$  contains the element  $B = (S_2 P + P^\perp)/\sqrt{2}$ . The results of [Ken11] imply that Theorem 5.4 of [DLP05] apply to the unital

weak operator closed algebra generated by any isometric tuple. Thus there is a net  $(B_\lambda)$  of elements in the unit ball of  $A_d$  such that  $w^*\text{-}\lim \pi_\phi(B_\lambda) = B$  in  $\mathcal{S}$ . It is easy to check that  $\|B\| = 1$  and  $\langle B(e + \xi_\emptyset)/\sqrt{2}, \xi_\emptyset \rangle = 1$ , so it follows that  $\|\phi\| = 1$ .

By the Hahn-Banach theorem, we can extend  $\phi$  to a functional on  $F_d^\infty$  with the same norm, which we continue to denote by  $\phi$ . Let  $\phi = \phi_a + \phi_s$  be the Lebesgue decomposition of  $\phi$  into absolutely continuous and singular parts as in Theorem 3.6. Then restricted to  $A_d$ , we can write

$$\begin{aligned}\phi_a(A) &= (\phi P^\perp)(A) = \langle \pi(A)\xi_\emptyset/\sqrt{2}, \xi_\emptyset \rangle, \quad A \in A_2, \\ \phi_s(A) &= (\phi P)(A) = \langle \pi(A)e/\sqrt{2}, \xi_\emptyset \rangle, \quad A \in A_2.\end{aligned}$$

Letting  $B$  be as above, an easy computation gives

$$\langle B\xi_\emptyset/\sqrt{2}, \xi_\emptyset \rangle = \langle Be/\sqrt{2}, \xi_\emptyset \rangle = 1/\sqrt{2}.$$

Arguing as before, this implies  $\|\phi_a\| \geq 1/\sqrt{2}$  and  $\|\phi_s\| \geq 1/\sqrt{2}$ . By Theorem 3.6, it follows that  $\|\phi_a\| + \|\phi_s\| = \sqrt{2}\|\phi\|$ .

#### 4. THE EXTENDED F. & M. RIESZ THEOREM

The results in this section can be viewed as noncommutative generalizations of the classical results referred to as the F. & M. Riesz theorem. As mentioned in the introduction, results of this kind have been established in different settings by Exel in [Exe90], and by Blecher and Labuschagne in [BL07]. In fact, Blecher and Labuschagne seem to have anticipated that an F. & M. Riesz-type theorem should hold for  $F_d^\infty$  (see the introduction of [BL07]).

**Theorem 4.1** (Extended F. & M. Riesz Theorem). *Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ , and let  $\phi = \phi_a + \phi_s$  be the Lebesgue decomposition of  $\phi$  into absolutely continuous and singular parts as in Theorem 3.6. Let  $\mathcal{I}$  be a two-sided ideal of  $F_d^\infty$ . If  $\phi$  is zero on  $\mathcal{I}$ , then  $\phi_a$  and  $\phi_s$  are both zero on  $\mathcal{I}$ .*

*Proof.* As in Section 2.3, there are vectors  $x$  and  $y$  in  $H_u$  such that

$$\phi(A) = \langle \pi_u(A)x, y \rangle, \quad \forall A \in F_d^\infty.$$

By Proposition 3.5 we can write  $\phi_a = \phi P_u^\perp$ , and by Proposition 3.3 we can write  $\phi_s = \phi P_u$ . If we identify  $F_d^\infty$  with its image  $\pi_u(F_d^\infty)$  in  $(F_d^\infty)^{**}$ , then

the functionals  $\phi$ ,  $\phi_a$  and  $\phi_s$  each have unique weak-\* continuous extensions to functionals on  $(F_d^\infty)^{**}$  with the same norm.

Let  $\mathcal{J}$  denote the ideal in  $(F_d^\infty)^{**}$  obtained by taking the weak-\* closure of  $\pi_u(\mathcal{I})$ . Since  $\phi$  is zero on  $\mathcal{I}$ , it is zero on  $\mathcal{J}$ . For  $A$  in  $\mathcal{I}$ ,  $\pi_u(A)P_u^\perp$  belongs to  $\mathcal{J}$ , which implies

$$0 = (\phi P_u^\perp)(A) = \phi_a(A).$$

Hence  $\phi_a$  is zero on  $\mathcal{I}$ , and it follows immediately that  $\phi_s$  is also zero on  $\mathcal{I}$ .  $\square$

**Corollary 4.2** (F. & M. Riesz Theorem). *Let  $\phi$  be a bounded linear functional on  $F_d^\infty$ . If  $\phi$  is zero on  $F_{d,0}^\infty$ , where  $F_{d,0}^\infty$  denotes the ideal of  $F_d^\infty$  generated by  $L_{Z_1}, \dots, L_{Z_d}$ , then  $\phi$  is absolutely continuous.*

*Proof.* Let  $\phi = \phi_a + \phi_s$  be the Lebesgue decomposition of  $\phi$  into absolutely continuous and singular parts as in Theorem 3.6. By Theorem 4.1,  $\phi_a$  and  $\phi_s$  are both zero on  $F_{d,0}^\infty$ . By Definition 3.1, if  $\phi_s$  is zero on  $F_{d,0}^\infty$ , it is necessarily zero on all of  $F_d^\infty$ . Hence  $\phi = \phi_a$  and  $\phi$  is absolutely continuous.  $\square$

## 5. QUOTIENT ALGEBRAS

For a weak-\* closed two-sided ideal  $\mathcal{I}$  of  $F_d^\infty$ , let  $\mathcal{A}_\mathcal{I}$  denote the quotient algebra  $F_d^\infty/\mathcal{I}$ .

**Definition 5.1.** Let  $\mathcal{I}$  be a weak-\* closed two-sided ideal of  $F_d^\infty$ , and let  $\phi$  be a bounded functional on  $\mathcal{A}_\mathcal{I}$ . Then

- (1)  $\phi$  is *absolutely continuous* if it is weak-\* continuous, and
- (2)  $\phi$  is *singular* if  $\|\phi\| = \|\phi^k\|$  for every  $k \geq 1$ , where  $\phi^k$  denotes the restriction of  $\phi$  to the ideal of  $\mathcal{A}_\mathcal{I}$  generated by  $\{\overline{L_{Z_w}} \mid |w| = k\}$ , where for a word  $w$  in  $\mathbb{F}_d^*$ ,  $\overline{L_{Z_w}}$  denotes the image in  $\mathcal{A}_\mathcal{I}$  of  $L_{Z_w}$ .

**Theorem 5.2** (Lebesgue decomposition for quotients of  $F_d^\infty$ ). *Let  $\mathcal{I}$  be a weak-\* closed two-sided ideal of  $F_d^\infty$ , and let  $\phi$  be a bounded linear functional on  $\mathcal{A}_\mathcal{I}$ . Then there are unique linear functionals  $\phi_a$  and  $\phi_s$  on  $\mathcal{A}_\mathcal{I}$  such that  $\phi = \phi_a + \phi_s$ , where  $\phi_a$  is absolutely continuous and  $\phi_s$  is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

*If  $d = 1$ , then the constant  $\sqrt{2}$  can be replaced with the constant 1.*

*Proof.* By basic functional analysis, we can lift the functional  $\phi$  to a functional  $\psi$  on  $F_d^\infty$  with the same norm. Let  $\psi = \psi_a + \psi_s$  be the Lebesgue

decomposition of  $\psi$  into absolutely continuous and singular parts as in Theorem 3.6. The functional  $\psi$  annihilates  $\mathcal{I}$ , so by Theorem 4.1, both  $\psi_a$  and  $\psi_s$  annihilate  $\mathcal{I}$ . Hence  $\psi_a$  and  $\psi_s$  induce functionals  $\phi_a$  and  $\phi_s$  on  $\mathcal{A}_{\mathcal{I}}$  respectively, with the same norm. Clearly  $\phi = \phi_a + \phi_s$ , and the inequality

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|$$

follows from the corresponding inequality in Theorem 3.6. The functional  $\phi_a$  is absolutely continuous since  $\psi_a$  is absolutely continuous on  $F_d^\infty$ . To see that  $\phi_s$  is singular, simply note that for every  $k \geq 1$ , the ideal  $(\mathcal{A}_{\mathcal{I},0})^k$  is the image in  $\mathcal{A}_{\mathcal{I}}$  of the ideal  $(F_{d,0}^\infty)^k$ .  $\square$

**Corollary 5.3** (Lebesgue decomposition for multiplier algebras). *Let  $\mathcal{A}$  be the multiplier algebra of a complete Nevanlinna-Pick space, and let  $\phi$  be a bounded linear functional on  $\mathcal{A}$ . Then there are unique linear functionals  $\phi_a$  and  $\phi_s$  on  $\mathcal{A}$  such that  $\phi = \phi_a + \phi_s$ , where  $\phi_a$  is absolutely continuous and  $\phi_s$  is singular, and such that*

$$\|\phi\| \leq \|\phi_a\| + \|\phi_s\| \leq \sqrt{2}\|\phi\|.$$

## 6. UNIQUENESS OF THE PREDUAL

Let  $X$  and  $Y$  be Banach spaces such that  $X^* = Y$ . Then  $X$  is said to be a predual for  $Y$ . Every predual  $X$  of  $Y$  naturally embeds into the dual space  $Y^*$ , and a subspace  $X$  of  $Y^*$  is a predual of  $Y$  if and only if it satisfies the following properties:

- (1) The subspace  $X$  norms  $Y$ , i.e.  $\sup\{|x(y)| \mid x \in X, \|x\| \leq 1\} = \|y\|$  for all  $y$  in  $Y$ , and
- (2) The closed unit ball of  $Y$  is compact in the  $\sigma(Y, X)$  topology.

The space  $Y$  is said to have a strongly unique predual if there is a unique subspace  $X$  of  $Y^*$  such that  $Y = X^*$ . For a survey on uniqueness results for preduals, we refer the reader to Godefroy's article [God89].

In the operator-theoretic setting, the results of Sakai [Sak56], Ando [And78] and Ueda [Ued09] mentioned in the introduction established that von Neumann algebras and maximal subdiagonal algebras have unique preduals. Ruan proved in [Rua92] that an operator algebra with a weak-\* dense subalgebra of compact operators has a unique predual, which applies to, for example, nest algebras and atomic CSL algebras. Effros, Ozawa and Ruan proved in [EOR01] that a  $W^*$ TRO (i.e. a corner of a von Neumann algebra) has a unique predual. More recently, in [DW11], Davidson and Wright

proved that a free semigroup algebra has a unique predual. Note that Davidson and Wright's result applies to  $F_d^\infty$ , but not to quotients of  $F_d^\infty$ .

The following definition was introduced by Godefroy and Talagrand in [GT80]. Recall that a (formal) series  $\sum_n y_n$  in a Banach space  $Y$  is weakly unconditionally Cauchy if  $\sum_n \phi(y_n) < \infty$  for every  $\phi \in Y^*$ .

**Definition 6.1.** A Banach space  $X$  has property (X) if, for every  $\phi \in X^{**} \setminus X$ , there is a weakly unconditionally Cauchy sequence  $(x_n)$  in  $X^*$  such that

$$\phi \left( \text{w}^* \lim_n \sum_{k=1}^n x_k \right) \neq \sum_{k=1}^{\infty} \phi(x_k).$$

One reason for the interest in property (X) is the following result of Godefroy and Talagrand from [GT80].

**Theorem 6.2** (Godefroy-Talagrand). *A Banach space  $X$  with property (X) is the unique predual of its dual.*

The following definition is closely related to the notion of an M-ideal in a Banach space (see [HWW93] for more information).

**Definition 6.3.** A Banach space  $X$  is L-embedded if there is a projection  $P$  on the bidual  $X^{**}$  with range  $X$  such that

$$\|x\| = \|Px\| + \|x - Px\|, \quad \forall x \in X^{**}.$$

The following result of Pfitzner from [Pfi07] implies that every separable L-embedded space has property (X), and hence that it is the unique predual of its dual.

**Theorem 6.4** (Pfitzner). *Separable L-embedded spaces have property (X).*

The results of Sakai, Ando and Ueda on decompositions of linear functionals imply that the preduals of von Neumann algebras and maximal subdiagonal algebras are L-embedded, and hence by Pfitzner's result from [Pfi07], that they are unique. However, Example 3.7 shows that quotients of  $F_d^\infty$  are not, in general, L-embedded, so we are unable to use Pfitzner's result. Instead, we give a direct proof that quotients of  $F_d^\infty$  have (strongly) unique preduals.

**Theorem 6.5.** *Let  $\mathcal{I}$  be a weak- $*$  closed two-sided ideal of  $F_d^\infty$ . Then the algebra  $\mathcal{A}_{\mathcal{I}}$  has a strongly unique predual.*



*Proof.* Suppose  $E$  is a predual for  $\mathcal{A}_{\mathcal{I}}$ , identified with a subspace of  $(\mathcal{A}_{\mathcal{I}})^*$ . By Theorem 5.2,

$$(\mathcal{A}_{\mathcal{I}})^* = (\mathcal{A}_{\mathcal{I}})_a^* \oplus (\mathcal{A}_{\mathcal{I}})_s^*,$$

where  $(\mathcal{A}_{\mathcal{I}})_a^*$  and  $(\mathcal{A}_{\mathcal{I}})_s^*$  denote the set of absolutely continuous and singular functionals on  $\mathcal{A}_{\mathcal{I}}$  respectively. We want to prove that  $E = (\mathcal{A}_{\mathcal{I}})_a^*$ .

Let  $\phi$  be a functional in  $E$ , and let  $\phi = \phi_a + \phi_s$  be the Lebesgue decomposition of  $\phi$  as in Theorem 5.2. We will prove that  $\phi_s = 0$ . Suppose to the contrary that  $\phi_s \neq 0$ . By basic functional analysis, we can lift the functional  $\phi$  to a functional  $\psi$  on  $F_d^\infty$  that is zero on  $\mathcal{I}$ . Let  $\psi = \psi_a + \psi_s$  be the Lebesgue decomposition of  $\psi$  as in Theorem 3.6. By Theorem 4.1,  $\psi_a$  and  $\psi_s$  are both zero on  $\mathcal{I}$ , and by construction they induce the functionals  $\phi_a$  and  $\phi_s$  respectively on the quotient  $\mathcal{A}_{\mathcal{I}}$ .

It follows from the results of [Ken11] that Theorem 5.4 of [DLP05] applies to the unital weak operator closed algebra generated by any isometric tuple. Thus there is a net  $(B_\lambda)$  of elements in the unit ball of  $F_d^\infty$  such that  $w^*\text{-}\lim \pi_u(B_\lambda) = P_u$  in  $(F_d^\infty)^{**}$ . Since the net  $(B_\lambda)$  is weak- $*$  convergent in  $(F_d^\infty)^{**}$ , it is weakly Cauchy in  $F_d^\infty$ . Since the closed unit ball of  $F_d^\infty$  is compact in the weak- $*$  topology, and in particular is complete, this implies that there is  $B$  in the closed unit ball of  $F_d^\infty$  such that  $w^*\text{-}\lim B_\lambda = B$  in  $F_d^\infty$ . For every weak- $*$  continuous functional  $\tau$  on  $F_d^\infty$ , Proposition 3.5 implies that

$$\tau(B) = \lim_\lambda \tau(B_\lambda) = (\tau P_u)(1) = 0.$$

Hence  $B = 0$ .

Let  $A$  be an element in the unit ball of  $F_d^\infty$  such that  $\psi_s(A) \neq 0$ . Since the net  $(B_\lambda)$  is weakly Cauchy in  $F_d^\infty$ , the image  $(\overline{B_\lambda})$  is weakly Cauchy in  $\mathcal{A}_{\mathcal{I}}$ . It follows that the net  $(\overline{AB_\lambda})$  is also weakly Cauchy in  $\mathcal{A}_{\mathcal{I}}$ . Since  $E$  is a predual of  $\mathcal{A}_{\mathcal{I}}$ , the closed unit ball of  $\mathcal{A}_{\mathcal{I}}$  is compact in the  $\sigma(\mathcal{A}_{\mathcal{I}}, E)$  topology, and in particular is complete. Thus, the net  $(\overline{AB_\lambda})$  converges in the  $\sigma(\mathcal{A}_{\mathcal{I}}, E)$  topology to an element  $C$  in the unit ball of  $\mathcal{A}_{\mathcal{I}}$ . By Proposition 3.3, we have

$$\phi(C) = \lim_\lambda \phi(\overline{AB_\lambda}) = \lim_\lambda \psi(AB_\lambda) = (\psi P_u)(A) = \psi_s(A) \neq 0,$$

so that  $C \neq 0$ . But since  $w^*\text{-}\lim B_\lambda = 0$  in  $F_d^\infty$ , it follows that  $w^*\text{-}\lim \overline{AB_\lambda} = 0$  in  $\mathcal{A}_{\mathcal{I}}$ . So for every  $\tau$  in  $(\mathcal{A}_{\mathcal{I}})_a^*$ , we necessarily have

$$\tau(C) = \lim_\lambda \tau(\overline{AB_\lambda}) = 0.$$

Since  $(\mathcal{A}_{\mathcal{I}})_a^*$  separates points, this implies that  $C = 0$ , which gives a contradiction. Thus  $\phi = \phi_a$ , meaning  $\phi$  is absolutely continuous.

Since  $\phi$  was arbitrary, it follows from above that every functional in  $E$  is absolutely continuous, i.e. that  $E$  is contained in  $(\mathcal{A}_{\mathcal{I}})_a^*$ . If it were the case that  $E \neq (\mathcal{A}_{\mathcal{I}})_a^*$ , then we could apply the Hahn-Banach theorem to separate  $E$  from  $(\mathcal{A}_{\mathcal{I}})_a^*$  with an element of  $\mathcal{A}_{\mathcal{I}}$ . But the fact that  $E$  is a predual of  $\mathcal{A}_{\mathcal{I}}$  means in particular it must norm  $\mathcal{A}_{\mathcal{I}}$ , so this is impossible. Therefore, we conclude that  $E = (\mathcal{A}_{\mathcal{I}})_a^*$ , and hence that  $(\mathcal{A}_{\mathcal{I}})_a^*$  is the unique predual of  $\mathcal{A}_{\mathcal{I}}$ .  $\square$

**Corollary 6.6.** *The multiplier algebra of every complete Nevanlinna-Pick space has a strongly unique predual.*

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